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Restrictions Imposed by the Optical Theorem on Exchanged Quantum Numbers*

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Crossing relations for particles of arbitrary spin are used to express the amplitudes for $A+B\to A+B$ in terms of amplitudes for the crossed reaction $A+\bar{A}\to B+\bar{B}$. Using the optical theorem, it is demonstrated that crossed amplitudes with positive parity, positive G parity, and positive signature cannot be negligible for forward scattering. Implications of this result for the spin dependences of high-energy scattering amplitudes are discussed.

I. INTRODUCTION

In a recent paper[1] it was demonstrated that in any elastic scattering reaction the total cross section is dominated by the exchange of systems with zero isotopic spin, in the sense that the contributions coming from the exchange of such systems cannot be negligible compared with those coming from the exchange of other isotopic spins. In the present work we demonstrate that, in the same sense, the total cross sections are dominated by the exchange of systems with even parity, even G parity, and even signature, as well as zero isotopic spin.

The proof depends, as in Ref. 1, on using the optical theorem for the forward scattering amplitude to impose positiveness conditions on the imaginary parts of certain amplitudes, and then considering the contributions to these imaginary parts of the various exchanged systems. One essential difference is the need to introduce amplitudes analytically continued out of the physical region, i.e., the need to invoke the concept of crossing. In the case of isotopic spin it was possible to work entirely in terms of exchange amplitudes defined by a unitary transformation on the elastic scattering amplitudes in states of definite total isotopic spin, without explicitly introducing the crossed reaction, though the results were equivalent to those expressed in terms of crossing. In the present case, however, the proof depends on using the fact that the exchange amplitudes we are concerned with are, in fact, analytically continued amplitudes for particle-antiparticle annihilation into a particle-antiparticle pair. We do not discuss the nature of this continuation here, but use results already given[2] for the crossing relation in the helicity representation.

II. CONSERVED QUANTUM NUMBERS

Consider the elastic scattering of a particle A having spin j_1 from a particle B having spin j_2. The process

$$A+B\to A+B \tag{1}$$

is described by a matrix $F(s,t)$ where s and t are the usual kinematic invariants defining the energy and angle. We shall also be concerned with the reactions associated with Eq. (1) by crossing

$$A+\bar{A}\to\bar{B}+B, \tag{2}$$

$$A+\bar{B}\to A+\bar{B}, \tag{3}$$

described respectively by matrices $M(s,t)$ and $N(s,t)$. For given s and t only one of the reactions is physically possible, the matrix elements of the others being defined by analytic continuation from the appropriate physical region.

The relation between the different matrices is best given in the helicity representation. For process (1) let λ_1 and λ_3 be the initial and final helicities of particle A and λ_2 and λ_4 those of particle B. We are then concerned with matrix elements $\langle\lambda_3\lambda_4|F(s,t)|\lambda_1\lambda_2\rangle$. Similar matrix elements with helicities μ_i and ν_i are defined for reactions (2) and (3), the notation being indicated in Fig. 1.

The relation between F and M in the helicity representation has been given in Ref. 2. For the case of forward scattering in process (1) the relation is

$$\langle\lambda_3\lambda_4|F(s,0)|\lambda_1\lambda_2\rangle=\sum_{\mu_i}(-1)^{\mu_2-\mu_4}\times d_{\mu_1\lambda_1}{}^{j_1}(\pi/2)d_{\mu_2\lambda_2}{}^{j_2}(\pi/2)d_{\mu_3\lambda_3}{}^{j_1}(\pi/2)d_{\mu_4\lambda_4}{}^{j_2}(\pi/2)\times\langle\mu_2\mu_4|M(s,0)|\mu_1\mu_3\rangle. \tag{4}$$

We begin by showing that only amplitudes with $P=G=(-1)^J$ contribute to the total cross section. Thus, if there is any scattering at all, the amplitudes

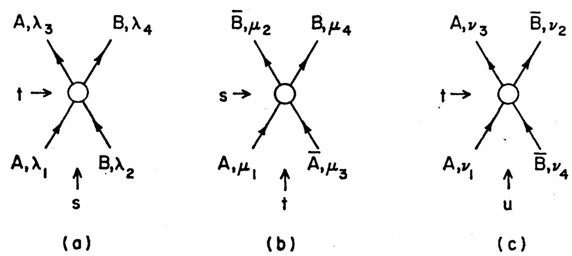

FIG. 1. The amplitude for reaction (a) is $\langle\lambda_3\lambda_4|F(s,t)|\lambda_1\lambda_2\rangle$; for (b) it is $\langle\mu_2\mu_4|M(s,t)|\mu_1\mu_3\rangle$; for (c) it is $\langle\nu_3\nu_2|N(s,t)|\nu_1\nu_4\rangle$.

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* Work performed under the auspices of the U. S. Atomic Energy Commission.
[1] L. L. Foldy and R. F. Peierls, Phys. Rev. **130**, 1585 (1963).
[2] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).

with $P=G=(-1)^J$ cannot be negligible. [Outside the physical region for reaction (2), $(-1)^J$ is simply a convenient notation for the signature of the system.] Remembering the result of Ref. 1, we shall assume $T=0$ and ignore charge henceforth. Then G is equivalent to charge conjugation.

First, continue $\langle\mu_2\mu_4|M(s,t)|\mu_1\mu_3\rangle$ to the physical region for reaction (2). Let $\psi_{p\lambda_1\lambda_3}$ denote the helicity state for $A\bar{A}$ as defined by Jacob and Wick.[3] The result of operating on $\psi_{p\lambda_1\lambda_3}$ with PG is given by

$$GP\psi_{p\lambda_1\lambda_3}=\psi_{p-\lambda_3-\lambda_1}. \tag{5}$$

From this relation, it follows that if we define the amplitudes

$$\begin{aligned}&\langle\lambda_2\lambda_4|M^{\pm}(s,t)|\lambda_1\lambda_3\rangle\\&=\tfrac{1}{2}\langle\lambda_2\lambda_4|M(s,t)|\lambda_1\lambda_3\rangle\pm\tfrac{1}{2}\langle\lambda_2\lambda_4|M(s,t)|-\lambda_3-\lambda_1\rangle,\end{aligned} \tag{6}$$

then $\langle\lambda_2\lambda_4|M^+(s,t)|\lambda_1\lambda_3\rangle$ is a pure $GP=+1$ amplitude and $\langle\lambda_2\lambda_4|M^-(s,t)|\lambda_1\lambda_3\rangle$ is a pure $GP=-1$ amplitude. The amplitudes $\langle\mu_2\mu_4|M^{\pm}(s,t)|\mu_1\mu_3\rangle$ may be expanded in a partial-wave series as in Eq. (31) of Ref. 3. For the particular amplitudes with zero total helicity $\langle\mu_2\mu_2|M^{\pm}(s,t)|\mu_1\mu_1\rangle$, it is seen that those with the upper sign get contributions only from terms with $(-1)^J=P$ while those with the lower sign get contributions only from terms with $(-1)^J=-P$. We can use these properties to attach a meaning to the symbol $(-1)^J$ even when $M(s,t)$ is continued back to the physical region for reaction (1) where the partial-wave expansion no longer converges.

The total cross section is given by

$$\sigma_{\text{tot}}=(4\pi/q\sqrt{s})\sum_{\lambda_1\lambda_2}\text{Im}\langle\lambda_1\lambda_2|F(s,0)|\lambda_1\lambda_2\rangle. \tag{7}$$

From Eqs. (4) and (6), it follows that

$$\begin{aligned}\sum_{\lambda_1\lambda_2}\langle\lambda_1\lambda_2|F(s,0)|\lambda_1\lambda_2\rangle&=\sum_{\mu_1\mu_2}\langle\mu_2\mu_2|M(s,0)|\mu_1\mu_1\rangle\\&=\sum_{\mu_1\mu_2}\langle\mu_2\mu_2|M^+(s,0)|\mu_1\mu_1\rangle.\end{aligned} \tag{8}$$

Thus, Eq. (8) and the results of the preceding paragraph imply that only amplitudes in the crossed channel with

$$G=P=(-1)^J \tag{9}$$

contribute to the total cross section, and hence cannot be negligible.

We must now demonstrate that the common value of these three quantum numbers is $+1$. To do this we consider the relation between M and N, the matrix for reaction (3). If we introduce the variable $\tilde{s}=2(m_A{}^2+m_B{}^2)-s-t$, then in the physical region for reaction (2) the interchange of the two final-state particles corresponds to replacing s by $\tilde{s}$. The G parity of a particle-antiparticle reaction determines whether the amplitude is even or odd under the interchange of particle and antiparticle in the final state[4]:

$$\begin{aligned}&\langle\mu_2\mu_4|M(s,t)|\mu_1\mu_3\rangle\\&\qquad=(-1)^{\mu_4-\mu_2}G\langle\mu_4\mu_2|M(\tilde{s},t)|\mu_1\mu_3\rangle.\end{aligned} \tag{10}$$

The crossing relation between M and N is the same as (4) with μ_4 and μ_2 interchanged and ν written for λ. If we now suppose that the only contribution to $\text{Im}M$ comes from odd G states then, using (7) and the corresponding result for N together with (10), we find

$$\begin{aligned}\sum_{\lambda_1\lambda_2}\text{Im}\langle\lambda_1\lambda_2|F(s,0)|\lambda_1\lambda_2\rangle&=\sum_{\mu_1\mu_2}\text{Im}\langle\mu_2\mu_2|M(s,0)|\mu_1\mu_1\rangle\\&=\sum_{\nu_1\nu_2}\text{Im}\langle\nu_1\nu_2|N(s,0)|\nu_1\nu_2\rangle\\&=-\sum_{\mu_1\mu_2}\text{Im}\langle\mu_2\mu_2|M(\tilde{s},0)|\mu_1\mu_1\rangle\\&=-\sum_{\nu_1\nu_2}\text{Im}\langle\nu_1\nu_2|N(\tilde{s},0)|\nu_1\nu_2\rangle.\end{aligned} \tag{11}$$

If $s>(m_A+m_B)^2$ then $\tilde{s}<(m_A-m_B)^2$ so that both $F(s,0)$ and $N(\tilde{s},0)$ correspond to physically possible processes.[5] Then (11) would imply that one had a negative total cross section, which is impossible. Hence the assumption of negative G must be wrong and the dominant terms must have even parity, G parity, and signature.[6]

III. DISCUSSION

The results of the last section can be summarized by stating that if an elastic scattering process is described in terms of the exchange of systems with various conserved quantum numbers, the contribution to the imaginary part of the forward scattering of terms with the quantum numbers of the vacuum (zero isotopic spin, even parity, even G parity, and even signature) cannot be negligible.

Such a description of an elastic scattering is useful in the discussion of very high-energy elastic processes. If it is assumed that different types of exchange have different asymptotic behaviors, then the asymptotically dominant contribution must correspond to the exchange of the quantum numbers of the vacuum. In particular, in the Regge-pole exchange description of high-energy elastic scattering, we can immediately conclude that the dominant, or Pomeranchuk trajectory *must* have the quantum numbers of the vacuum.

This assignment is, of course, the usual one; however, the usual argument is to observe that this is the correct

[3] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

[4] L. Van Hove, Phys. Letters **7**, 76 (1963).

[5] Notice that the quantities $\langle\nu_3\nu_2|N(s,t)|\nu_1\nu_4\rangle$ are defined so that they correspond to the amplitudes for the physical process (3) when $s<(m_A-m_B)^2-t$, $t<0$.

[6] This result depends on the relative phase of $F(s,t)$ and $N(s,t)$ assumed in (11). This phase is not determined by the work of Ref. 2, since only transformation properties are used there. In (11), the relative phase is assumed to be zero; this is the case in all models known to us.

assignment to ensure the validity of the Pomeranchuk-Okun hypothesis about the behavior of high-energy elastic scattering, whereas in the present case the only principles invoked have been the optical theorem and sufficient analyticity of the helicity amplitudes to lead to the crossing relation (4). While it would be very surprising if any other set of quantum numbers were to be singled out in this way, it is by no means obvious *a priori* that such a restriction should follow from such weak assumptions.

The set of quantum numbers specified above is the maximum number of conserved quantum numbers which can be specified.[7] However, it is not always necessary nor always sufficient to uniquely specify the reaction in the crossed channel. If either j_1 or j_2 are equal to zero, for example, the parity and signature are equivalent; if, further, the zero spin particle has a definite G parity, then the isotopic spin determines all the other quantum numbers. On the other hand, the next simplest case possible is the elastic scattering of two spin-$\frac{1}{2}$ particles; in this case the specification of all the stated quantum numbers leaves three independent amplitudes for the crossed reaction (the three triplet, $J\neq L$ amplitudes) and for higher spins the ambiguities increase. Indeed, in the forward direction, the exchange of a system with $G=P$ and $T=0$ contributes only to the diagonal amplitudes $\langle\lambda_1\lambda_2|F(s,0)|\lambda_1\lambda_2\rangle$. If this is coupled with the symmetry condition (which is valid for any parity conserving interaction)

$$\langle\lambda_1\lambda_2|F(s,t)|\lambda_1\lambda_2\rangle=\langle-\lambda_1-\lambda_2|F(s,t)|-\lambda_1-\lambda_2\rangle, \quad (12)$$

one obtains what seems to be the maximum number of relations between the amplitudes which can be obtained on the basis of the present assumptions.

It is interesting to see what further relations can be obtained by making additional assumptions. For example, what must we assume in order to obtain the result that $F(s,0)$ is a multiple of the unit matrix? It is clear from (4) that we must have

$$\langle\mu_2\mu_4|M(s,0)|\mu_1\mu_3\rangle=\text{const.}\,\delta_{\mu_2\mu_4}\delta_{\mu_1\mu_3}. \quad (13)$$

However, this result is obtained if we make the weaker assumption:

$$\langle\mu_2\mu_4|M(s,0)|\mu_1\mu_3\rangle=M_{\mu_2\mu_1}\delta_{\mu_2\mu_4}\delta_{\mu_1\mu_3}. \quad (13')$$

This, is, in a sense, a natural assumption to make since we know from (8) that such terms must be present if there is to be any scattering at all. Further, it is consistent with everything we have shown thus far. Then (4) becomes, with

$$\langle\lambda_3\lambda_4|F(s,0)|\lambda_1\lambda_2\rangle=\delta_{\lambda_2\lambda_4}\delta_{\lambda_1\lambda_3}F_{\lambda_2\lambda_1},$$

$$\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}F_{\lambda_2\lambda_1}=\sum_{\mu_1\mu_2}M_{\mu_2\mu_1}$$
$$\times d_{\mu_2\lambda_2}{}^{j_1}(\pi/2)d_{\mu_2\lambda_4}{}^{j_2}(\pi/2)d_{\mu_1\lambda_1}{}^{j_1}(\pi/2)d_{\mu_1\lambda_3}{}^{j_1}(\pi/2). \quad (14)$$

Multiplying by $d_{\mu_2'\lambda_2}{}^{j_2}(\pi/2)d_{\mu_1'\lambda_1}{}^{j_1}(\pi/2)$ and summing on λ_1 and λ_2 we obtain

$$d_{\mu_2'\lambda_4}{}^{j_2}(\pi/2)d_{\mu_1'\lambda_3}{}^{j_1}(\pi/2)F_{\lambda_4\lambda_3}$$
$$=M_{\mu_2'\mu_1'}d_{\mu_2'\lambda_4}{}^{j_2}(\pi/2)d_{\mu_1'\lambda_3}{}^{j_1}(\pi/2); \quad (15)$$

hence $F_{\lambda_4\lambda_3}=M_{\mu_2'\mu_1'}$ for any λ_i, μ_i' and the result follows. Instead of (13′), one could assume that the amplitudes M factorize:

$$\langle\mu_2\mu_4|M(s,t)|\mu_1\mu_3\rangle=\alpha_{\mu_1\mu_3}\beta_{\mu_2\mu_4}, \quad (16)$$

a standard assumption of Regge-pole calculations. This leads to additional relations among the $\langle\lambda_1\lambda_2|\times F(s,0)|\lambda_1\lambda_2\rangle$ (they factorize), but does not lead to the full simplicity of spin independence, except for j_1, $j_2\leq\frac{1}{2}$.

One could ask other questions. For example, the result that certain amplitudes must not be negligible can be made more precise by deriving certain equalities and inequalities among the amplitudes M. This can be done, following Foldy,[8] by noting that the optical theorem requires that the diagonal elements of F be positive (or zero) in any spin basis. As an example, for NN scattering one obtains the condition

$$\langle\tfrac{1}{2}\tfrac{1}{2}|M(s,0)|-\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\geq\langle\tfrac{1}{2}\,-\tfrac{1}{2}|M(s,0)|-\tfrac{1}{2}\tfrac{1}{2}\rangle, \quad (17)$$

among others.

In summary, the optical theorem requires that if the total cross section is dominated by the exchange of a system with definite quantum numbers, then the quantum numbers must be those of the vacuum. In general, however, the assumption that the scattering is actually dominated by the exchange of such a system is not sufficient to imply spin independence of the forward amplitudes.

[7] Of course, there are other sorts of quantum numbers, such as charge or strangeness, which are determined by the assumption of elastic scattering. The isospin 0 assignment of Ref. 1 has been extended to demonstrate the dominance of the identity representation of any internal symmetry group. See D. Amati, L. L. Foldy, A. Stanghellini, L. Van Hove, CERN Report 7774/TH. 393 (unpublished).

[8] L. L. Foldy (to be published).